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Letter

# Exact Lyapunov exponents of the generalized Boole transformations

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The generalized Boole transformations have rich behavior ranging from the *mixing* phase with the Cauchy invariant measure to the *dissipative* phase through the *infinite ergodic* phase with the Lebesgue measure. In this letter, by giving the proof of mixing property for  $0 < \alpha < 1$ , we show an *analytic* formula of the Lyapunov exponents  $\lambda$  that are explicitly parameterized in terms of the parameter  $\alpha$  of the generalized Boole transformations for the whole region  $\alpha > 0$  and bridge these three phases *continuously*. We find different scale behaviors of the Lyapunov exponent near  $\alpha = 1$  using an analytic formula with the parameter  $\alpha$ . In particular, for  $0 < \alpha < 1$ , we then prove the existence of the extremely sensitive dependence of Lyapunov exponents, where the absolute values of the derivative of the Lyapunov exponents with respect to the parameter  $\alpha$  diverge to infinity in the limits of  $\alpha \rightarrow 0$  and  $\alpha \rightarrow 1$ . This result shows the computational complexity of the numerical simulations of the Lyapunov exponents near  $\alpha \simeq 0, 1$ .  
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Subject Index     A30, A33

1. *Introduction: Generalized Boole transformations* The Lyapunov exponent has been extensively studied for characterizing chaotic systems and plays an important and essential role as a chaos indicator, quantitatively characterizing the degree of chaoticity. However, one can find it difficult to obtain an analytic formula for Lyapunov exponents for given chaotic systems in general. There are a few analytic formulas for ergodic invariant measures [1]. The analytic parametric density functions of the ergodic invariant measure can be used for analytically estimating Lyapunov exponents for such specific systems. However, for most cases, the analytic result cannot be extended for the whole region of parameter space. Consequently, we are forced to use numerical simulations to research the parameter dependence of Lyapunov exponents. In the case of the logistic map, Huberman and Rudnick [2] found a scaling behavior of the Lyapunov exponent from zero to positive; Pomeau and Manneville [3] found a similar behavior. Both of these works use numerical simulations.

Here, we consider the Boole transformation [4] given by

$$x_{n+1} = Tx_n = x_n - \frac{1}{x_n}, \quad (1)$$

which is known to preserve the Lebesgue measure  $dx$  as an invariant measure [5]. The Lebesgue measure  $dx$  is an infinite measure. Thus, we cannot normalize the invariant measure as a probability measure.

One can generalize the Boole transformation as a two-parameter map family, as Aaronson [6] treats the following generalized Boole transformations, which are defined as

$$x_{n+1} = T_{\alpha, \beta} x_n = \alpha x_n - \frac{\beta}{x_n}, \quad \alpha > 0, \beta > 0, \quad (2)$$

preserving the Cauchy measure  $\mu(dx)$  for  $0 < \alpha < 1$  given by

$$\mu(dx) = \frac{\sqrt{\beta(1-\alpha)}}{\pi[x^2(1-\alpha) + \beta]} dx. \quad (3)$$

As a subset of the generalized Boole transformations, the map at  $\alpha = \beta = 1/2$  corresponds to the exactly solvable chaos map [1] given by Ref. [7]:

$$x_{n+1} = \frac{1}{2} \left( x_n - \frac{1}{x_n} \right). \quad (4)$$

One of the authors showed that this map can be expressed by the double-angle formula of the cot function with a Lyapunov exponent  $\lambda = \log 2$  and one can use Eq. (4) in order to generate general Lévy stable distributions via their superposition [7].

The generalized Boole transformations have standard forms that do not depend on  $\beta$ . That is, by substituting  $x_n$  with  $\sqrt{\frac{\beta}{\alpha}} y_n$ , Eq. (2) is replaced by

$$\sqrt{\frac{\beta}{\alpha}} y_{n+1} = \alpha \sqrt{\frac{\beta}{\alpha}} y_n - \beta \sqrt{\frac{\alpha}{\beta}} \frac{1}{y_n},$$

which is nothing but

$$y_{n+1} = \alpha \left( y_n - \frac{1}{y_n} \right). \quad (5)$$

The transformation  $P_{\alpha, \beta} : x_n \mapsto \sqrt{\frac{\beta}{\alpha}} y_n$  is a one-to-one diffeomorphism. Then the transformations of Eq. (5) have the same Lyapunov exponent for fixed  $\alpha$  with this topological conjugacy relation. Consequently, the generalized Boole transformations have the same Lyapunov exponents, which are independent of  $\beta$ .

## 2. Analytic formula of Lyapunov exponents for the generalized Boole transformations

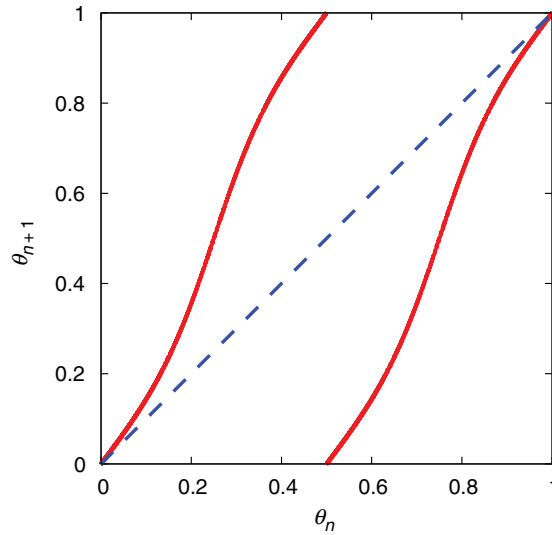
The ergodicity of similar kinds of generalized Boole transformations was shown by Kempermann [8] and Letac [9]. We prove Theorem 1, concerning the mixing property of the generalized Boole transformations for  $0 < \alpha < 1$ , as follows.

**THEOREM 1** The generalized Boole transformations in  $0 < \alpha < 1$  have a mixing property.

Thus, we can prove the ergodicity with respect to the Cauchy measure in Eq. (3) for  $0 < \alpha < 1$ .

*Proof* We obtain the relation of Eq. (5) by substituting  $x_n$  with  $-\cot \pi \theta_n$  as

$$\begin{aligned} \cot \pi \theta_{n+1} &= 2\alpha \cot 2\pi \theta_n, \\ \text{where } \theta_{n+1} &= \bar{T}_\alpha(\theta_n) = \frac{1}{\pi} \operatorname{arccot} [2\alpha \cot 2\pi \theta_n]. \end{aligned} \quad (6)$$



**Fig. 1.** Solid lines correspond to the transformation  $\bar{T}_{0.75}$ , which is topologically conjugate with the generalized Boole transformation  $T_{\alpha=0.75, \beta=0.75}$ , and the dashed line corresponds to  $\theta_{n+1} = \theta_n$ . The transformations  $\bar{T}_\alpha$  monotonically increase in the interval  $[0, 0.5)$  or  $[0.5, 1)$ .

The transformations

$$\bar{T}_\alpha : [0, 1) \rightarrow [0, 1) \quad (7)$$

have a topological conjugacy relation with Eq. (5). Consider the following cylinder sets of  $\bar{T}_\alpha^{-k}[0, 1)$  on which  $\bar{T}_\alpha^k$  is surjective to  $[0, 1)$ , satisfying

$$I_{j,k} = [v_{j,k}, v_{j+1,k}), \quad v_{j,k} < v_{j+1,k}, \quad \text{for } 0 \leq j \leq 2^k - 1, \quad (8)$$

$$v_{0,k} = 0 \text{ and } v_{2^k,k} = 1, \quad (9)$$

$$\bar{T}_\alpha^k I_{j,k} = [0, 1), \quad (10)$$

$$\mu(I_{j,k}) = \frac{1}{2^k}, \quad (11)$$

(see Figs 1 and 2).

For any measurable set  $A$  and  $\bar{T}_\alpha$ -invariant measure  $\mu$ , consider, for arbitrary  $n(\geq k)$ ,

$$\mu(\bar{T}_\alpha^{-n} A \cap B), \quad (12)$$

where it is enough to assume that  $B$  is  $[v_{j,k}, v_{j+1,k})$  without loss of generality. Here,  $\mu(B) = \frac{1}{2^k}$  and the number of intervals of  $\bar{T}_\alpha^{-n} A$  that are included by  $B$  is  $2^{n-k}$ . Thus, the measure of  $\bar{T}_\alpha^{-n} A$  on each interval  $I_{j,k}$ , whose measure  $\mu(I_{j,k})$  is *equidistributed*, is  $2^{-n} \mu(A)$ . Then, for arbitrary  $n(\geq k)$ ,

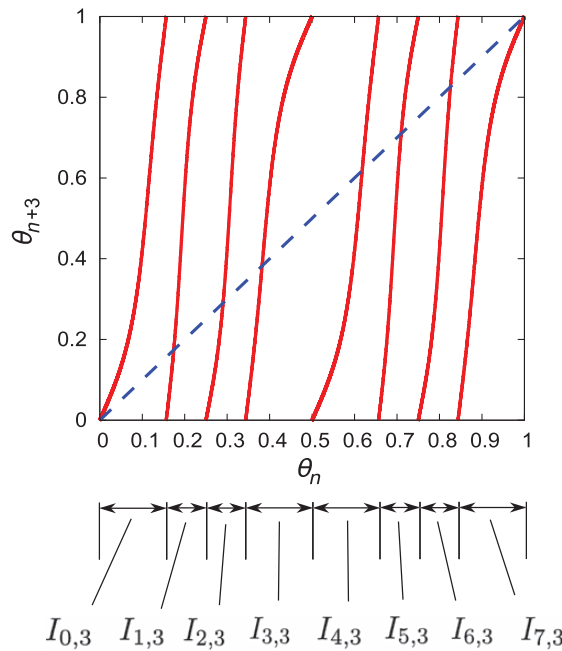
$$\mu(\bar{T}_\alpha^{-n} A \cap B) = 2^{n-k} \cdot (2^{-n} \mu(A)) = \mu(A) \cdot \mu(B). \quad (13)$$

Thus, the dynamical system  $([0, 1), \bar{T}_\alpha, \mu)$  has a mixing property according to Ref. [16]. Therefore, the generalized Boole transformations for  $0 < \alpha < 1$  have a mixing property. ■

Now we prove the following theorem.

**THEOREM 2** The generalized Boole transformations for  $0 < \alpha < 1$  have Lyapunov exponents given by

$$\lambda(\alpha, \beta) = \log \left( 1 + 2\sqrt{\alpha(1-\alpha)} \right).$$



**Fig. 2.** Solid lines correspond to the transformation  $\bar{T}_{0.75}^3$ , which is topologically conjugate with the generalized Boole transformation  $T_{\alpha=0.75, \beta=0.75}^3$ , and the dashed line corresponds to  $\theta_{n+3} = \theta_n$ . Here, the measure of each  $I_{j,3}$  ( $0 \leq j \leq 7$ ) is equidistributed.

Note that the Lyapunov exponents do not depend on the parameter  $\beta$  but purely depend on the parameter  $\alpha$ .

*Proof* By the ergodicity, we can calculate the Lyapunov exponents of the generalized Boole transformations as

$$\lambda(\alpha, \beta) = \int_{-\infty}^{\infty} \log \left| \alpha + \frac{\beta}{x^2} \right| \frac{\sqrt{\beta(1-\alpha)}}{\pi [x^2(1-\alpha) + \beta]} dx.$$

Here, we substitute  $\sqrt{\alpha/\beta}x$  with  $y$  and obtain:

$$\lambda(\alpha, \beta) = \frac{1}{\pi} \sqrt{\frac{\alpha}{1-\alpha}} \int_{-\infty}^{\infty} \log \left| \alpha + \frac{\alpha}{y^2} \right| \frac{dy}{y^2 + \left( \sqrt{\frac{\alpha}{1-\alpha}} \right)^2},$$

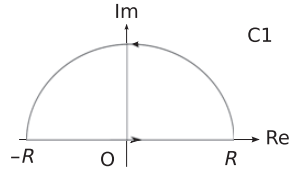
$$\log \left| \alpha + \frac{\alpha}{y^2} \right| = \log \alpha + \log |1 + iy| + \log |1 - iy| - 2 \log |y|.$$

Let us substitute  $p$  for  $\sqrt{\frac{\alpha}{1-\alpha}}$  and put  $I_1, I_2, I_3$ , and  $I_4$  as

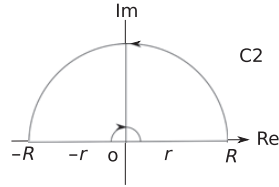
$$\lambda(\alpha, \beta) = I_1 + I_2 + I_3 + I_4,$$

$$I_1 = \frac{p}{\pi} \int_{-\infty}^{\infty} \frac{\log \alpha}{y^2 + p^2} dy,$$

$$I_2 = \frac{p}{\pi} \int_{-\infty}^{\infty} \frac{\log |1 + iy|}{y^2 + p^2} dy,$$



**Fig. 3.** The integral path of  $I_1$ ,  $I_2$ , and  $I_3$ .



**Fig. 4.** The integral path of  $I_4$ .

$$I_3 = \frac{p}{\pi} \int_{-\infty}^{\infty} \frac{\log |1 - iy|}{y^2 + p^2} dy,$$

$$I_4 = -\frac{2p}{\pi} \int_{-\infty}^{\infty} \frac{\log |y|}{y^2 + p^2} dy.$$

Evaluate  $I_1$ . We consider a function  $f_1(z)$  defined by

$$f_1(z) = \frac{\log \alpha}{z + ip},$$

which is regular on the upper half-plane. Thus we can integrate along the path C1 in Fig. 3. Then we have the relations

$$I_1 = \log \alpha \quad (14)$$

as  $R \rightarrow \infty$ .

Next, as for  $I_3$ , we consider a function  $f_3(z)$  defined by

$$f_3(z) = \frac{\log |1 - iz|}{z + ip},$$

which is regular on the upper half-plane. Thus we can integrate  $f_3(z)$  along the path C1 in Fig. 3. We get:

$$I_3 = \log |1 + p| \quad (15)$$

as  $R \rightarrow \infty$ .

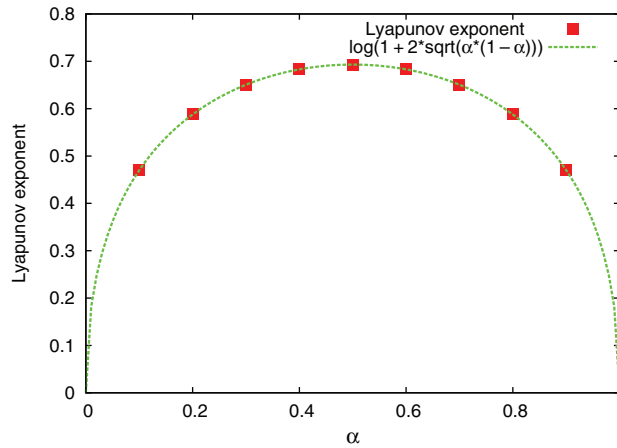
In terms of evaluating  $I_2$ , by applying a transformation as  $y \rightarrow -y$  we have the relation  $I_2 = I_3$ .

As for  $I_4$ , we consider a function  $f_4(z)$  defined by

$$f_4(z) = \frac{\log |z|}{z + ip}, \quad (16)$$

which is regular along the path C2 in Fig. 4 and on the inner part of C2. Then, by integrating  $f_4(z)$  along C2, we obtain

$$I_4 = -2 \log p \quad (17)$$



**Fig. 5.** The numerical results of the finite Lyapunov exponents and analytic solution of the generalized Boole transformation.

as  $r \rightarrow 0$ ,  $R \rightarrow \infty$ . As for  $r \rightarrow 0$ , we substitute  $z$  with  $e^{i\theta}$  and have the relation

$$\left| \int_{\pi}^0 \frac{\log r + 2\pi}{r^2 e^{2i\theta} - p^2} i r e^{i\theta} d\theta \right| \leq \int_0^{\pi} \frac{|\log r + 2\pi|}{|r^2 e^{2i\theta} - p^2|} r d\theta.$$

Then we set  $r < p/2$  and obtain:

$$\frac{1}{|r^2 e^{2i\theta} + p^2|} < \frac{1}{|p|^2 - r^2} < \frac{4}{3} \frac{1}{|p|^2}.$$

Using these facts, we have

$$\begin{aligned} \int_0^{\pi} \frac{|\log r + 2\pi|}{|r^2 e^{2i\theta} - p^2|} r d\theta &< \int_0^{\pi} \frac{4}{3} \frac{|\log r + 2\pi|}{|p|^2} r d\theta, \\ &\leq \frac{4\pi r}{3} \frac{|\log r + 2\pi|}{|p|^2} \rightarrow 0 \text{ as } r \rightarrow 0. \end{aligned}$$

Therefore, we obtain the analytic formula for the Lyapunov exponents for the generalized Boole transformations  $\lambda(\alpha, \beta)$  as

$$\begin{aligned} \lambda(\alpha, \beta) &= \log \alpha + \log |1 + p| + \log |1 + p| - 2 \log p, \\ &= \log \left( 1 + 2\sqrt{\alpha(1-\alpha)} \right). \end{aligned} \quad (18)$$

■

The Lyapunov exponents do not depend on  $\beta$ , so that we can change the notation from  $\lambda(\alpha, \beta)$  to  $\lambda(\alpha)$ . The invariant measure  $\mu(dx)$  is Cauchy and smooth with respect to the Lebesgue measure. Thus, according to the Pesin identity [10,11], we can calculate the Kolmogorov–Sinai (KS) entropy  $h(\alpha)$  analytically as

$$h(\alpha) = \lambda(\alpha). \quad (19)$$

Figure 5 shows the comparison results of our numerical estimate of the Lyapunov exponents for the generalized Boole transformations with  $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ , and  $0.9$  using the long double floating format and our analytic formula for the Lyapunov exponents. The initial condition of computation is  $x_0 = 9.3$ . The coincidence is remarkable, as shown in Fig. 5.

Equation (18) converges to zero in the limit of  $\alpha \rightarrow 0$  and  $\alpha \rightarrow 1$ . However, different phenomena occur for  $\alpha = 0$  and  $\alpha = 1$ , respectively. In the case of  $\alpha = 0$ , every orbit is periodic because

$$T_{\alpha,\beta}^2 x = x.$$

On the other hand, at  $\alpha = 1$ , *subexponential chaos* [12,13] occurs. Then, in calculation of the finite Lyapunov exponents, the value does not converge to a constant but converges in distribution.

### 3. Lyapunov exponents for $\alpha > 1$ Here, we prove another theorem.

**THEOREM 3** The generalized Boole transformations for  $\alpha > 1$  have Lyapunov exponents given by

$$\lambda(\alpha, \beta) = \log \alpha.$$

*Proof* In the range of  $\alpha > 1$ , the generalized Boole transformations have two equilibrium points at  $x_{\pm} = \pm\sqrt{\frac{\beta}{\alpha-1}}$ . Two of them are unstable since the slopes of the linear inclinations are both  $\frac{dT_{\alpha,\beta}x}{dx}(x_{\pm}) = 2\alpha - 1 > 1$ . Now by substituting  $x_n$  with  $\frac{1}{z_n}$ , we obtain another form of the generalized Boole transformations as

$$z_{n+1} = \frac{z_n}{\alpha - \beta z_n^2}. \quad (20)$$

The equilibrium points of Eq. (20) are  $z = 0, \pm\sqrt{\frac{\alpha-1}{\beta}}$ . Two points  $z = \pm\sqrt{\frac{\alpha-1}{\beta}}$  are unstable. The equilibrium point at  $z = 0$  is stable and attractive because

$$\left| \frac{d}{dz} \left( \frac{z}{\alpha - \beta z^2} \right) \right|_{z=0} < 1. \quad (21)$$

Thus, for almost all initial points, it holds that

$$\lim_{n \rightarrow \infty} |x_n| = \infty. \quad (22)$$

Hence, the Lyapunov exponents of the generalized Boole transformations for  $\alpha > 1$  are given by taking the logarithmic function of inclination in the limit of  $|x| \rightarrow \infty$  as

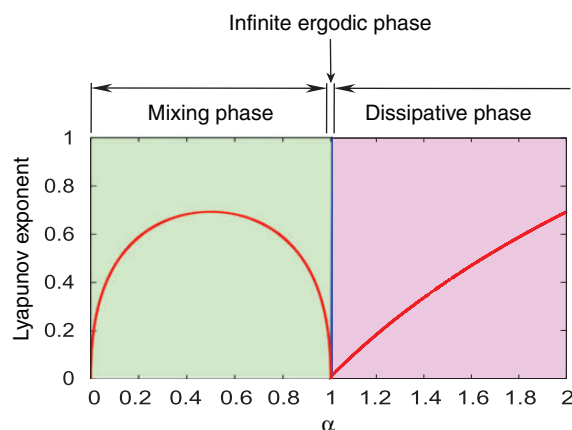
$$\lambda(\alpha, \beta) = \log \alpha. \quad (23)$$

■

The complete behavior of the Lyapunov exponent as a function of the parameter  $\alpha$  is shown in Fig. 6. According to Ref. [6], the generalized Boole transformations for  $\alpha > 1$  are *dissipative* in the sense that, for almost all initial points,  $|x_n| \rightarrow \infty$  as  $n \rightarrow \infty$ .

Table 1 summarizes the classification of the phase of systems by the nature of the invariant measure. We can say the system is in the *mixing phase* for  $0 < \alpha < 1$ , *infinite ergodic phase* for  $\alpha = 1$ , and *dissipative phase* at  $\alpha > 1$ .





**Fig. 6.** The classification of the phase of the generalized Boole transformation systems in the parameter space  $\alpha$  via the analytic Lyapunov exponents.

**Table 1.** The classification of the phase of dynamical systems, invariant measure, and Lyapunov exponents in terms of the parameter  $\alpha$ .

	$0 < \alpha < 1$	$\alpha = 1$	$\alpha > 1$
Phase	Mixing phase	Infinite ergodic phase	Dissipative phase
Invariant measure $\mu(dx)$	$\frac{\sqrt{\beta(1-\alpha)}}{\pi [x^2(1-\alpha) + \beta]} dx$	$dx$	
Lyapunov exponents	$\log(1 + 2\sqrt{\alpha(1-\alpha)})$	0 ( <i>subexponential chaos</i> )	$\log \alpha$

The standard Cauchy density function  $\rho_\gamma(x)$  is defined by a scale parameter  $\gamma$ :

$$\rho_\gamma(x) = \frac{\gamma}{\pi (x^2 + \gamma^2)}. \quad (24)$$

According to the probability preservation relation [15], the density functions  $\rho_\gamma(x)$  and  $\rho_{\gamma'}(y)$  satisfy the relation

$$\rho_{\gamma'}(y)dy = \sum_{x=T_{\alpha,\beta}^{-1}y} \rho_\gamma(x)dx. \quad (25)$$

In the case of  $\alpha = \beta = \frac{1}{2}$  [15], the scale parameter  $\gamma$  of the Cauchy density function of the invariant measure is updated in accordance with

$$\gamma' = \frac{1}{2} \left( \gamma + \frac{1}{\gamma} \right). \quad (26)$$

Then,  $\gamma$  converges to unity. Similarly, by the probability preservation relation, the scale parameter of the density function of the generalized Boole transformation is updated in accordance with

$$\gamma' = \alpha\gamma + \frac{\beta}{\gamma}, \quad (27)$$

and it converges to the fixed point  $\gamma^* = \sqrt{\frac{\beta}{1-\alpha}}$ .

As we make  $\alpha$  close to unity from below, the scale parameter  $\gamma$  converges to infinity and the density function of the invariant measure becomes the Lebesgue measure  $dx$ .

For  $\alpha > 1$ , the fixed point of the scale parameter  $\gamma^*$  must be an imaginary number. This fact shows that, for  $\alpha > 1$ , the density function of the invariant measure can *never* be a Cauchy distribution.

Compared with the Ant-Lion map [12,14], whose Lyapunov exponent diverges to infinity at the fixed point, the generalized Boole transformations for  $\alpha > 1$  have a positive Lyapunov exponent  $\log \alpha$  at the attractive fixed point  $z = 0$ .

4. *Scaling behavior at  $\alpha = 0, 1$*  With a parametric analytic formula for the Lyapunov exponents, we can investigate a parametric dependence. The first derivative of the Lyapunov exponents with respect to  $\alpha$  is given by

$$\frac{d\lambda}{d\alpha} = \begin{cases} \frac{1-2\alpha}{\sqrt{\alpha(1-\alpha)}(1+2\sqrt{\alpha(1-\alpha)})}, & 0 < \alpha < 1, \\ \frac{1}{\alpha}, & \alpha > 1. \end{cases} \quad (28)$$

Remarkably, it diverges in the limits of  $\alpha \rightarrow 0, \alpha \rightarrow 1 - 0$  as

$$\left| \lim_{t \rightarrow 0} \frac{d\lambda}{d\alpha} \Big|_{\alpha=t} \right| = \infty, \quad \left| \lim_{t \rightarrow 1-0} \frac{d\lambda}{d\alpha} \Big|_{\alpha=t} \right| = \infty. \quad (29)$$

This is similar to the well known critical phenomenon that the derivative of magnetization diverges at the critical temperature. This means that a *slight* modification of the parameter  $\alpha$  toward unity (the Boole transformations) from below has a *large* effect on the value of the Lyapunov exponent at the edge of  $0 < \alpha < 1$ . Furthermore, we can say that it is *extremely* difficult to numerically obtain the Lyapunov exponents near  $\alpha = 1$  and  $0 < \alpha < 1$  because, for  $\alpha = 1$ , *subexponential chaos* [12,13] occurs and we cannot determine a Lyapunov exponent for a finite calculation.

According to Eq. (5), the generalized Boole transformation at the limit of  $\alpha \rightarrow 1$  corresponds to the Boole transformation, such that a transition behavior between chaotic behavior and *subexponential* behavior [13] can be observed at  $\alpha = 1$ .

Consider the scaling behavior at  $\alpha = 0, 1$ . When  $\alpha \simeq 0$  or “ $\alpha \simeq 1$  and  $0 < \alpha < 1$ ”, the relation  $0 < 2\sqrt{\alpha(1-\alpha)} \ll 1$  holds. Thus, we obtain a Taylor expansion of Eq. (18) as

$$\lambda(\alpha) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left( 2\sqrt{\alpha(1-\alpha)} \right)^n. \quad (30)$$

Thus, the first approximation is given by the first term of the Taylor expansion:

$$\lambda \simeq 2\sqrt{\alpha(1-\alpha)}, \quad (31)$$

$$\simeq \begin{cases} 2\sqrt{\alpha}, & \text{for } \alpha \simeq 0, \\ 2\sqrt{1-\alpha}, & \text{for } \alpha \simeq 1 \text{ and } 0 < \alpha < 1. \end{cases} \quad (32)$$

Then, the scaling behavior of the Lyapunov exponents near  $\alpha = 0, 1$  is shown as

$$\lambda \simeq \begin{cases} 2\alpha^{\frac{1}{2}}, & \text{for } \alpha \simeq 0, \\ 2(1-\alpha)^{\frac{1}{2}}, & \text{for } \alpha \simeq 1 \text{ and } 0 < \alpha < 1, \\ \alpha - 1, & \text{for } \alpha \simeq 1 \text{ and } \alpha > 1. \end{cases} \quad (33)$$

In this paper, we define a critical exponent as a power exponent of a Lyapunov exponent. A critical exponent  $\delta$  is defined by

$$\lambda \simeq C|r - r_c|^\delta, \quad (34)$$

where  $r$  is a parameter,  $r_c$  is a critical point, and  $C$  is a constant. The  $\alpha$ -dependence of  $\delta$  is summarized in Table 2. Therefore, both of the critical exponents  $\delta$  at the edge of  $0 < \alpha < 1$  of the scaling behavior

**Table 2.** The difference between types of intermittency and critical exponents of the generalized Boole transformation.

	$\alpha \rightarrow 0 + 0$	$\alpha \rightarrow 1 - 0$	$\alpha \rightarrow 1 + 0$
Intermittency	<i>Type 3</i>	<i>Type 1</i>	
Critical exponent	$\frac{1}{2}$	$\frac{1}{2}$	1

of Lyapunov exponents are  $\frac{1}{2}$ . The Floquet multiplier for the fixed point for  $\alpha = 0$  is  $-1$ ; that for  $\alpha = 1$  is  $1$ . Thus, our analytic results agree with the prediction by Pomeau and Manneville [3] that critical exponents of Lyapunov exponents of *Type 1* (where the Floquet multiplier is unity) and *Type 3* (where the Floquet multiplier is  $-1$ ) intermittency are  $\frac{1}{2}$ , respectively. The critical exponent in the limit of  $\alpha \rightarrow 1 + 0$  is different from that of  $\alpha \rightarrow 1 - 0$ . This remarkable difference in the critical exponents of Lyapunov exponents is caused by the difference between the mixing (ergodic) phase for  $0 < \alpha \leq 1$  and the dissipative (nonergodic) phase for  $\alpha > 1$ .

**5. Conclusion** We obtain an analytic formula for Lyapunov exponents for the generalized Boole transformations for the full range of parameters and prove the mixing of the map for  $0 < \alpha < 1$ . We also obtain the KS entropy by using Pesin's formula. As a result, we can explicitly modulate the Lyapunov exponent  $\lambda$  in the range of  $\lambda \geq 0$  by changing the parameter  $\alpha > 0$ . In addition, the absolute values of the derivative  $d\lambda/d\alpha$  at  $\alpha = 0$  to  $\alpha = 1 - 0$  are proven to be *infinite*. Then, it is analytically shown to be extremely difficult to distinguish values of the Lyapunov exponent near  $\alpha = 0$  or  $\alpha = 1$  because of such strong parameter dependence. The scaling behavior of our analytic Lyapunov exponents of the Boole transformations is found to be consistent with Pomeau and Manneville's numerical results for *Type 1* and *Type 3* intermittency. Thus, we expect that our analytic analysis will be useful for further investigation of physically chaotic systems.

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